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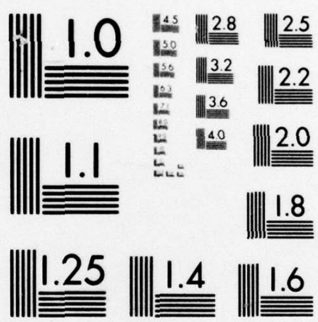
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QUADRATIC INTERPOLATION IS RISKY

Stephen M. Robinson

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

This brief note points out that the method of quadratic interpolation, which has been recommended in the literature for minimizing a function of one variable, can be very undependable. In particular, unless the function being minimized is itself quadratic, the method may break down no matter how close to the minimizer one starts.

AMS(MOS) Subject Classifications: 65K05, 90C30.

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SIGNIFICANCE AND EXPLANATION

In many practical applications we have to find numerically the minimum or maximum value of a function of one variable: an example might be a statistical estimation problem in which we try to find the value of a parameter which maximizes the likelihood function.

A simple method which has been recommended in the literature is to find the function values at three distinct points, then fit a quadratic function (a parabola) to these. The maximum or minimum value of this quadratic function is then found, together with the point at which it is attained, and this point is used together with two of the original three points to repeat the process.

This note points out that unless the function being minimized is itself quadratic, the above algorithm may break down (the new point may be one of the original three even though none of these minimizes the function). Even if the method does not break down, it may provide inaccurate answers because of the effects of roundoff error. Thus, it could be unwise to use this method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

QUADRATIC INTERPOLATION IS RISKY

Stephen M. Robinson

Let f be a function from $\mathbb{R}$ into itself. In this note we study some properties of the following algorithm for finding a minimum of f if one exists:

QUADRATIC INTERPOLATION ALGORITHM: Given three distinct points $x_0 < x_1 < x_2$, and the values of f at these points, find the unique polynomial P of degree no greater than 2 which agrees with f at x_0, x_1 and x_2. If P has a minimum, let x_3 be a minimizer of P; replace one of x_0, x_1, x_2 by x_3 to obtain three new points; *stop or repeat the process.*

The replacement may be done according to any one of several rules: discard the "oldest" point, discard the point with the greatest function value, etc. Also, various stopping rules may be applied to halt the algorithm at an appropriate step. For example, see the discussion in [1, pp. 605-606], in which the rule used would specify, for our formulation, that the three retained points $z_0 < z_1 < z_2$ satisfy $f(z_1) < \min\{f(z_0), f(z_2)\}$. It is stated there that the use of this rule will produce a sequence of points infinitely many of which are within any given tolerance of a minimizer of f (the authors of [1] phrase their discussion in terms of maximization, but this has no effect on the development). As the function illustrated in their Fig. 13.21 [1, p. 605] is unimodal, one might suppose that they mean for their statement to be interpreted in terms of such a function. No discussion is given in [1] of the rate at which the asserted convergence occurs, but this question is examined in [2, §2.3], where an argument is made with a view to establishing such a rate.

In what follows, we shall show that unless the function f is itself quadratic, this method may break down, no matter how close to a minimizer of f the initial points are chosen: that is, the new point obtained from the algorithm may be one of the current three points even though none of these is the minimizer. Next we discuss briefly the numerical problems to which the method is subject even when it does not break down. It will be apparent from these considerations that the quadratic interpolation method is unlikely to be the algorithm of choice in most applications.

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In the following proposition, we suppose that we are dealing with a unimodal function f: i.e., one for which there is a point $x_* \in \mathbb{R}$ with the property that for x_1, x_2 in the domain of f, if $x_1 < x_2 \leqq x_*$ then $f(x_1) > f(x_2)$, and if $x_* \leqq x_1 < x_2$ then $f(x_1) < f(x_2)$. Evidently x_* is then the unique minimizer of f.

PROPOSITION: Let $x_* \in \mathbb{R}$ and let $\varepsilon > 0$. Suppose f is a continuous, unimodal function from the interval $I := [x_* - \varepsilon, x_* + \varepsilon]$ to $\mathbb{R}$ with minimizer x_*. Unless f agrees with some quadratic function on I, there exist points $x_0 < x_1 < x_2$ in the interval I, with $x_1 \neq x_*$, such that the quadratic function Q interpolating f at these three points has x_1 as its unique minimizer.

Thus, unless f is actually quadratic near its minimizer x_*, the algorithm may break down.

PROOF: We distinguish two cases. For the first case, suppose that there is some $\eta \in (0,\varepsilon]$ with $f(x_* - \eta) \neq f(x_* + \eta)$. For convenience assume that $f(x_* - \eta) < f(x_* + \eta)$, and let $x_0 := x_* - \eta$. As f is continuous and strictly increasing on $[x_*, x_* + \eta]$, there is a point $x_2 \in (x_*, x_* + \eta)$ with $f(x_2) = f(x_0)$ (x_2 lies in the open interval because $f(x_*) < f(x_0) < f(x_* + \eta)$). Now let $x_1 := \frac{1}{2}(x_0 + x_2)$; evidently $x_0 < x_1 < x_* < x_2$, so $f(x_1) < f(x_0) = f(x_2)$ by unimodality. It is easy to verify that the unique quadratic function Q agreeing with f at x_0, x_1 and x_2 attains its minimum only at x_1. The argument for the case $f(x_* - \eta) > f(x_* + \eta)$ is similar, and this completes the proof of the first case.

In the second case, we know that $f(x_* - \eta) = f(x_* + \eta)$ for each $\eta \in (0,\varepsilon]$ but that f does not agree with any quadratic on I. For brevity, write $x_L := x_* - \varepsilon$ and $x_R := x_* + \varepsilon$. Of course, $f(x_L) = f(x_R) > f(x_*)$. Now consider the family of quadratic functions in x defined by

$$Q(y;x) := f(x_L)\left(\frac{x-y}{x_L-y}\right)^2 + f(y)\left[1 - \left(\frac{x-y}{x_L-y}\right)^2\right] ,$$

where y is a given point in (x_L, x_R). The function $Q(y;\cdot)$ interpolates f at x_L and at y, and it has a unique minimizer at y.

Since f does not agree with any quadratic on $[x_L, x_R]$, it certainly does not agree with $Q(x_*;\cdot)$ there, and so there is some point $z \in [x_L, x_R]$ with $f(z) \neq Q(x_*; z)$. A quick computation shows that $Q(x_*; x_R) = f(x_R)$, so actually $z \in (x_L, x_R)$, and by symmetry we can suppose that $z \in (x_*, x_R)$. Assume first that $f(z) < Q(x_*; z)$. Evidently the function of y given by $Q(y;z)$ is continuous for $y \in (x_L, x_R)$, so we can find $x_1 \in (x_*, z)$ with $Q(x_1; z) > f(z)$. We have $f(x_1) < f(x_L)$ by unimodality and $f(x_R) = f(x_L)$ by symmetry, so

$$Q(x_1; x_R) - f(x_R) = [f(x_1) - f(x_L)]\left[1 - \left(\frac{x_R - x_1}{x_L - x_1}\right)^2\right] < 0 \quad .$$

Thus the continuous function $Q(x_1;\cdot) - f(\cdot)$ is positive at z but negative at x_R, so there is some $x_2 \in (z, x_R)$ with $Q(x_1; x_2) = f(x_2)$. Thus, with $x_0 := x_L$ we find that $x_0 < x_1 < x_2$, that $Q(x_1;\cdot)$ interpolates f at these three points and has its unique minimizer at x_1, but that $x_1 > x_*$.

If $f(z) > Q(x_*; z)$ then we choose $x_1 \in (x_L, x_*)$ with $Q(x_1; z) < f(z)$; we then find that $Q(x_1; x_R) > f(x_R)$ so that a similar argument applies. This completes the proof.

We have shown that the quadratic interpolation method may break down; even if it does not do so, numerical difficulties are likely to prevent one from finding a very accurate estimate of the minimizer. These difficulties are well known to any experienced numerical analyst. They are shared by all methods using only function values, and are illustrated by the problem of minimizing $1 + x^2$: it is easy to see that if one works to $2N$ significant figures, the computed function value for any point in the interval $[-2 \times 10^{-N}, 2 \times 10^{-N}]$ will be 1. Thus one cannot expect to be able to determine the minimizer to within more than about half the number of significant digits carried by the machine. This loss of significance does not occur with methods which attempt to find a zero of the first derivative (e.g., Newton's method or the secant method), since near a nondegenerate minimizer the absolute value of the first derivative will be roughly proportional to the distance to the minimizer instead of to the square of that distance. Thus, when derivatives are available it is generally wise to make use of them. Even when only function values can be computed, the use of a reliable search method (e.g., golden section search) seems likely to be safer than quadratic interpolation because search algorithms are not subject to the breakdown property discussed above.

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[1] S. P. Bradley, A. C. Hax and T. L. Magnanti, Applied Mathematical Programming (Addison-Wesley, Reading, Mass., 1977).

[2] J. Kowalik and M. R. Osborne, Methods for Unconstrained Optimization Problems (American Elsevier, New York, 1968).

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